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STATE-SPACE APPROXIMATION OF MULTI-INPUT MULTI-OUTPUT SYSTEMS WITH STOCHASTIC EXOGENOUS INPUTS

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Abstract—Instead of the usual AR(MA)X- or VAR (vector autoregressive) modelling, procedures will be described to obtain approximate balanced state-space models and steady-state Kalman filters with prewhitened inputs. These state-space models and Kalman filters can be used for prediction and also for control whenever the output and input variables are target and control variables respectively.

1. INTRODUCTION

Econometricians and system theorists differ in the way they model dynamical multivariable input–output time series. In econometrics the so-called AR(MA)X—and VAR (vector autoregressive)—models seem to be popular, whereas in system theory—since the pioneering work of Kalman—the so-called state-space realization procedures received a lot of attention as efficient methods to model the dynamical behavior of the system by means of the introduction of the so-called state vector which accumulates information from the past (inputs) in as much it is relevant for the future (outputs). As stated by Åström (1984) the state vector can be seen as an abstraction of, for example, the state of a particle, which is determined by its position and momentum. The future motion of the particle is solely determined by its present state (and external forces). There is a close connection between state-space and Markov modelling (e.g. Picci, 1982). An advantage of state-space modelling is that the dynamical behavior of the multivariable input–output time series can be modelled efficiently with a parsimony in the description of the relation future outputs–past inputs. The Kalman filter can be used directly for prediction and because in this paper the input series are prewhitened, no *a priori* prediction for the input variables are needed in the Kalman filter.

System theory provides further model reduction algorithms for the (balanced) state-space model. Also LQG-control algorithms are directly applicable whenever the input and output variables are control and target variables respectively. In Section 2 the various models will be introduced and balanced state-space realization procedures discussed. The problem of approximating a given (balanced) state-space model by a lower order model yet maintaining the input–output behavior "as much as possible" will also be discussed. In Section 3 a procedure will be given to obtain approximate balanced state-space realizations from finite data samples and in Section 4 the suggested procedures will be applied to two multi-input, single-output macroeconomic equations. The paper is ended by a conclusions section.

2. BALANCED STATE-SPACE REALIZATIONS AND APPROXIMATIONS

Consider the following infinite distributed lag model (convolution) or final form equation obtained from distributed lag models, i.e.

$$y_t = E(L)u_t + \delta_t,$$

where y_t is a p-dimensional output (endogenous) vector; u_t a q-dimensional input (exogenous) vector and $\{\delta_t\}$ Gaussian white-noise with covariance V_δ. The matrix polynoom $E(L)$ in the lag operator L is given by

$$E(L) = \sum_{i=0}^{\infty} E_i L^i$$

and is assumed to be stable. Assume the input series $\{u_t\}$ to be a stationary, stochastic process which admits an ARMA-representation, i.e.

$$F(L)u_t = G(L)\tilde{\eta}_t,$$

where it is assumed that $\det(F(L)) \neq 0$. The input innovation sequence $\{\tilde{\eta}_t\}$ is assumed to be a Gaussian white-noise process with covariance $V_{\tilde{\eta}}$ of full rank which can be factorized as $V_{\tilde{\eta}} = F_{\tilde{\eta}}F'_{\tilde{\eta}}$ where $F_{\tilde{\eta}}^{-1}$ exists. Substituting

$$u_t = F^{-1}(L)G(L)\tilde{\eta}_t$$

into the infinite distributed lag model yields

$$y_t = M(L)\eta_t + \delta_t, \tag{1}$$

where

$$\eta_t = F_{\tilde{\eta}}^{-1}\tilde{\eta}_t$$

is standard Gaussian white-noise ($\eta_t \sim \mathrm{NID}(0, I)$) and where the matrix polynomial

$$M(L) = E(L)F^{-1}(L)G(L)F_{\tilde{\eta}}^{-1} = \sum_{i=0}^{\infty} m_i L^i$$

is assumed to be stable. The output series $\{y_t\}$ is a stationary process with zero mean and as input the innovation sequence $\{\eta_t\}$ of the input series $\{u_t\}$. Relating at time t the future output to the past inputs we have

$$\begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \cdot \\ \vdots \end{bmatrix} = \begin{bmatrix} m_1 & m_2 & \cdots \\ m_2 & m_3 & \cdots \\ m_3 & \cdot & \cdots \\ \vdots & & \end{bmatrix} \begin{bmatrix} \eta_t \\ \eta_{t-1} \\ \eta_{t-2} \\ \vdots \end{bmatrix} + \begin{bmatrix} m_0 & 0 & \cdots \\ m_1 & m_0 & \cdots \\ \cdot & \cdot & \\ \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \eta_{t+1} \\ \eta_{t+2} \\ \cdot \\ \vdots \end{bmatrix} + \begin{bmatrix} \delta_{t+1} \\ \delta_{t+2} \\ \cdot \\ \vdots \end{bmatrix}$$

or

$$y_t^+ = H\eta_t^- + T\eta_t^+ + \delta_t^+,$$

where H is a so-called Hankel matrix and T a Toeplitz matrix. The *covariance* of the future output y_t^+ and past input innovation sequence η_t^- is given by the Hankel matrix, i.e.

$$\epsilon\{y_t^+(\eta_t^-)'\} = H\epsilon\{\eta_t^-(\eta_t^-)'\} + T\epsilon\{\eta_t^+(\eta_t^-)'\} + \epsilon\{\delta_t^+(\eta_t^-)'\} = H$$

where $\epsilon\{\ \}$ denotes the expectation operator. Assume the rank of the Hankel matrix to be finite, i.e. $r(H) = n < \infty$, which implies that the future outputs depend on a finite past sequence of inputs. Consider the following so-called state-space realization of the infinite distributed lag model (1):

$$\begin{aligned} x_{t+1} &= Ax_t + B\eta_t, \\ y_t &= Cx_t + D\eta_t + \delta_t, \end{aligned} \tag{2}$$

where the state vector is given by x_t with $\dim(x_t) = r(H) = n$.

Substituting

$$x_t = (IL^{-1} - A)^{-1}B\eta_t$$

into the second equation of (2) yields the transfer function model

$$y_t = C(I - AL)^{-1}B\eta_{t-1} + D\eta_t + \delta_t, \tag{3}$$

from which it can be seen that

$$m_i = CA^{i-1}B, \quad i \geqslant 1 \quad \text{and } D = m_0,$$

which in turn implies that the Hankel matrix H can be factorized as

$$H = QP = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} (B \quad AB \quad A^2B \ldots),$$

where Q and P are respectively the so-called (extended) observability and controllability matrix, provided that A is stable. The state-space realization (2) is however, not unique. It can be shown that every linear (coordinate) transformation of the state vector, i.e.

$$\tilde{x}_t = Tx_t,$$

leads to the same transfer function model (3). It is said that all state-space models with $(TAT^{-1}, TB, CT^{-1}, D)$ where T is a regular transformation matrix are T-equivalent with (A, B, C, D), the state-space model (2). In this paper one particular realization will be considered namely the (internally) balanced realization as introduced by Moore (1981), which is suitable for model reduction. In this paper we follow a simple balanced and approximation procedure as suggested by Kung and Lin (1981); see also Silverman and Bettayeb (1980). According to Kung and Lin, the Hankel matrix H can be written as the following singular value decomposition (SVD):

$$H = U\Sigma V' = \sum_{i=1}^{n} \sigma_i u_i v_i'$$

where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$ with singular values $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$, u_i and v_i are infinite dimensional vectors with $u_i'v_j = v_i'v_j = \delta_{ij}$ where δ_{ij} is the Kronecker delta. The matrices U and V are $U = (u_1, \ldots, u_n)$ and $V = (v_1, \ldots, v_n)$, respectively. From the foregoing we have seen that H can be factorized as $H = QP$. Let $Q = U\Sigma^{1/2}$ and $P = \Sigma^{1/2}V'$ then $H = QP = U\Sigma V'$.

Denote by "$\uparrow$" the shift-up operator defined as

$$QA = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} A = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \end{bmatrix} = Q^{\uparrow}. \tag{4}$$

From equation (4) we have as solution for A, denoted by A_b,

$$A_b = Q^{\#}Q^{\uparrow} = \Sigma^{1/2}U'(U\Sigma^{1/2})^{\uparrow},$$

where

$$Q^{\#} = (Q'Q)^{-1}Q' = \Sigma^{-1/2}U'$$

is a pseudo-inverse of Q. Because the first q columns of $P = \Sigma^{1/2}V'$ constitute the matrix B, where q is the dimension of the input vector we have as solution for B, B_b = the first q columns of $P = \Sigma^{1/2}V'$ and for the matrix C, C_b = the first p rows of $Q = U\Sigma^{1/2}$.

This particular realization (state-space model) with (A_b, B_b, C_b, D) where $D = m_0$ is called a balanced realization. See for a discussion and interpretation of "balancing" Moore (1981), Aoki (1987) or Otter (1987).

Suppose that some of the singular values are close to zero. The question arises whether the balanced state-space model with the n-dimensional state vector may be approximated by a lower order balanced state-space model with a (say) k-dimensional state-vector where $k < n$, yet maintaining the input/output behavior of the system "as much as possible", measured by some norm. Suppose the singular values $\sigma_{k+1} \approx \sigma_{k+2} \approx \cdots \approx 0$. Partition H as

$$H = (U_1, U_2)\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} V_1' \\ V_2' \end{bmatrix}$$

with $\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_k)$, $\Sigma_2 = \text{diag}(\sigma_{k+1}, \ldots, \sigma_n)$, $U_1 = (u_1, \ldots, u_k)$, $U_2 = (u_{k+1}, \ldots, u_n)$,

$V_1 = (v_1, \ldots, v_k)$ and $V_2 = (v_{k+1}, \ldots, v_k)$. An (non-unique) approximant of H which—in general—does not preserve the Hankel structure is given by

$$L_1 = U_1 \Sigma_1 V_1 = \sum_{i=1}^{k} \sigma_i u_i v_i',$$

where L_1 minimizes $\|H - \tilde{L}\|_s$ over all matrices $\tilde{L}$ of rank k and where $\|H - L_1\|_s = \sigma_{k+1}$. $\|X\|_s$ denotes the maximum eigenvalue of $X'X$. Decompose

$$L_1 = U_1 \Sigma_1 V_1' = Q_1 P_1$$

with $Q_1 = U_1 \Sigma_1^{1/2}$ and $P_1 = \Sigma_1^{1/2} V_1'$ then a reduced order (quasi-balanced) state-space model with dim $(\tilde{x}_t) = k$ is given by

$$\tilde{x}_{t+1} = A_r \tilde{x}_t + B_r \eta_t,$$
$$y_t = C_r \tilde{x}_t + D\eta_t + \delta_t,$$

where

$$A_r = Q_1^{\#} Q_1^{\uparrow} = \Sigma_1^{1/2} U_1' (U_1 \Sigma_1^{1/2})^{\uparrow}$$
$$B_r = \text{first } q \text{ columns of } P_1 = \Sigma_1^{1/2} V_1' \text{ and}$$
$$C_r = \text{first } p \text{ rows of } Q_1 = U_1 \Sigma_1^{1/2} \text{ and } D = m_0.$$

Now it can be shown (Silverman, 1980) that $A_r = A_{11}$, $B_r = B_1$ and $C_r = C_1$ where A_{11}, B_1 and C_1 are sub-matrices of the original balanced state-space model, i.e.

$$A_b = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{k\text{-columns}}^{k\text{-rows}}, \quad B_b = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^{k\text{-rows}}, \quad C_b = \underbrace{(C_1 \quad C_2)}_{k\text{-columns}}$$

and hence all reduced order models are nested in the balanced state-space model with matrices (A_b, B_b, C_b, D). Further it is shown by Pernebo and Silverman (1982) that A_{11} is stable if A_b is stable.

The degree of approximation can be measured by Gelfand and Yaglom's information content by considering the singular values of H to be the canonical correlation coefficients (see for a discussion Desai and Pal, 1982, or Otter, 1987). In order to predict with the balanced state-space model or a reduced order model given by considering a sub-model of the balanced realization, we have to estimate the sequence $\{x_t\}$ from the input and output series $\{\eta_t\}$ and $\{y_t\}$, which can be done optimally by using the (steady-state) Kalman filter. According to Bertsekas (1976) we have that because the pair (A_b, C_b) is observable and (A_b, B_b) is controllable "in the large" the Kalman filter reaches its steady-state given by the equations

$$\hat{x}_{t+1|t} = A_b \hat{x}_{t|t-1} + \beta_b \eta_{t+1} + A_b K e_t, \tag{5a}$$

with initial condition $\hat{x}_0 = \mu$,

$$y_t = C_b \hat{x}_{t|t-1} + D\eta_t + e_t, \tag{5b}$$

where K is the steady-state Kalman gain given by

$$K = V_{\hat{x}} C_b' (C_b' V_{\hat{x}} C_b + \omega_\delta = V_\delta)^{-1}$$

and $V_{\hat{x}}$ is the state *a priori* error-covariance matrix of the Kalman MSE predictor $\hat{x}_{t+1|t}$ for all t satisfying the algebraic Riccati equation

$$V_{\hat{x}} = A_b [V_{\hat{x}} - V_{\hat{x}} C_b' (C_b V_{\hat{x}}' C_b' + V_\delta)^{-1} C_b V_{\hat{x}}] A_b'.$$

The output prediction-error is

$$e_t = y_t - \hat{y}_{t|t-1} = y_t - C_b \hat{x}_{t|t-1} - D\eta_t,$$

where it is assumed that we know the input innovation sequence $\{\eta_t\}$ in advance. Methods to solve the algebraic Riccati equation are given, for example, in Bertsekas (1976).

3. QUASI-BALANCED STATE-SPACE APPROXIMATIONS FROM SAMPLES

Until now we discussed the balanced realization procedures for the infinite distributed lag model where it was implicity assumed that the matrix polynomials $E(L)$, $F(L)$, $G(L)$ together with the input innovation sequence $\{\eta_t\}$ and covariance V_η were known. In this section we relax this assumption and give the following procedure to obtain quasi-balanced approximations from finite data records.

Step 1

Prewhiten the input sequence $\{u_t, t = 0, \dots, T\}$ by fitting q ARMA models collected in the following multivariate ARMA description:

$$\hat{F}(L)u_t = \hat{G}(L)\tilde{\eta}_t,$$

along the lines of Box and Jenkins (1970) or Goodwin and Payne (1977). The sequence $\{\tilde{\eta}_t\}$ is the sequence of prediction-errors (realizations of η_t) which is a Gaussian white noise process with

$$\tilde{\eta}_t = u_t - \hat{u}_{t|t-1},$$

where the prediction

$$\hat{u}_{t|t-1} = -\hat{F}_1 u_{t-1} - \cdots - \hat{F}_p u_{t-p} + \hat{G}_1\tilde{\eta}_{t-1} + \hat{G}_q\tilde{\eta}_{t-q}.$$

The standard white-noise input prediction-error

$$\hat{\eta}_t = \hat{F}_\eta^{-1}\tilde{\eta}_t,$$

where the estimated covariance matrix $\hat{V}_\eta$ is factorized as

$$\hat{V}_\eta = \hat{F}_\eta\hat{F}_\eta'.$$

Step 2

Estimate the matrices $\{m_i\}$ from the model

$$y_t = M(L)\hat{\eta}_t + \delta_t, \quad t = 1, \dots, T$$

by fitting an ARX model (matrix fraction description), i.e.

$$\hat{C}(L)y_t = \hat{D}(L)\hat{\eta}_t + \delta_t,$$

with

$$\hat{M}(L) = \hat{C}^{-1}(L)\hat{D}(L) = \sum_{i=1}^{\infty} \hat{m}_i L^i$$

(e.g. Goodwin and Payne, 1977; Ljung and Söderström, 1983).

Form the finite $Np \times Nq$ Hankel matrix

$$H_n = \begin{bmatrix} \hat{m}_1 & \cdots & \hat{m}_N \\ \vdots & & \\ \hat{m}_n & \cdots & \hat{m}_{2N-1} \end{bmatrix},$$

where N is an integer. The estimated finite Hankel matrix can be decomposed by means of the SVD with $p \leqslant q$ as

$$H_N = U(\Sigma\colon 0)\begin{bmatrix} V_1' \\ V_2' \end{bmatrix} = U\Sigma V_1' = \sum_{i=1}^{pN} \sigma_i u_u v_i',$$

where $U = (u_1, \dots, u_{pN})$ is an orthogonal matrix; $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_{pN})$ with singular values $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{pN} > 0$ and $V = (V_1 V_2) = (V_1, \dots, V_{pN}, V_{pN+1}, \dots, V_{qN})$ is a $(qN \times qN)$ orthogonal matrix. Because the Hankel matrix consists of estimates $\hat{m}_i$, $i = 1, \dots, (2N-1)$ obtained from noisy data it is to be expected that $r(H_N) = pN$, i.e. full row rank. A balanced realization with state-vector $\tilde{x}_t$ and $\dim(\tilde{x}_t) = pN$ is given by

$$\tilde{x}_{t+1} = \hat{A}_b \hat{x}_t + \hat{B}_b \hat{\eta}_t,$$
$$y_t = \hat{C}_b \tilde{x}_t + \hat{D}\hat{\eta}_t + \delta_t,$$

with $\hat{A}_b = Q^{\#} Q^{\uparrow}$; $\hat{B}_b =$ first q columns of P, $\hat{C}_b =$ first p rows of Q and $\hat{D} = \hat{m}_0$. Here $Q^{\#} = (Q'Q)^{-1}Q'$ is the pseudo-inverse of Q with $Q = U\Sigma^{1/2}$ and $P = \Sigma^{1/2}V_1'$. The shifted matrix

$$Q^{\uparrow} = \begin{bmatrix} Q_2 \\ \cdots \\ 0 \end{bmatrix} \text{ with } Q = \begin{bmatrix} Q_1 \\ \cdots \\ Q_2 \end{bmatrix},$$

where Q_1 consists of the first p rows of Q.

Suppose the last $(pN - k)$ singular values "close to zero", i.e. $\sigma_{k+1} \approx \sigma_{k+2} \approx \sigma_{pN} \approx 0$. A kth order approximation of the balanced realization is given by

$$x_{t+1} = \hat{A}_{11} x_t + \hat{B}_1 \hat{\eta}_t,$$
$$y_t = \hat{C}_1 x_t + \hat{D}\hat{\eta}_t + \delta_t, \tag{6}$$

where x_t is the k-dimensional state vector and $\hat{A}_b$, $\hat{B}_b$ and $\hat{C}_b$ are partitioned as

$$\hat{A}_b = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{B}_b = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{C}_b = (\hat{C}_1 \quad \hat{C}_2),$$

with $\hat{A}_{11}$ a $(k \times k)$ matrix, $\hat{B}_1$ a $(k \times q)$ matrix and $\hat{C}_1$ a $(p \times k)$ matrix. The steady-state Kalman filter for the kth order approximation (6) is

$$\hat{x}_{t+1|t} = \hat{A}_{11} \hat{x}_{t|t-1} + \hat{B}_1 \hat{\eta}_t + \hat{A}_{11} K e_t, \text{ initial value } \hat{x}_0 = \mu,$$
$$y_t = \hat{C}_1 \hat{x}_{t|t-1} + \hat{D}\hat{\eta}_t + e_t, \quad t = 1, \ldots, T,$$

where the output prediction-error $e_t = y_t - \hat{y}_{t|t-1}$ with output prediction $\hat{y}_{t|t-1} = \hat{C}_1 \hat{x}_{t|t-1}$ because the best prediction for the Gaussian white-noise input prediction $\hat{\eta}_{t|t-1} = 0$. The Kalman gain is denoted by K.

Step 3

(Re-)estimate or refine the parameters of $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, K, \hat{D}, \mu)$ collected in the s-dimensional parameter vector θ by minimizing the prediction-error criterion

$$J_T(\theta) := \log \det D_T(\theta),$$

where the sample prediction-error covariance is

$$D_T(\theta) = T^{-1} \sum_{j=1}^{T} e_j(\theta) e_j'(\theta),$$

where $e_j(\theta) = y_j - y_{j|j-1}(\theta)$ are the output prediction-errors generated by the Kalman filter with parameter vector θ. See for this so-called prediction-error estimation Ljung and Söderström (1983).

4. APPROXIMATED QUASI-BALANCED REALIZATIONS OF TWO MACROECONOMIC EQUATIONS

For preliminary analysis of the suggested procedures we considered two behavioral equations of a small macroeconomic model describing the private consumption and gross investments in the Netherlands in the period 1952–1983 (see Dietzenbacher, 1985).

Private consumption

$$\underset{}{c} = \underset{(0.054)}{0{,}525 L_{B-1/2}} + \underset{(0.041)}{0{,}143 NL_{B-1/2}} + \underset{(0.018)}{0{,}063 c_c}$$

Private gross investments

$$i_m = 1{,}430v'_{-1/2} + 1{,}722K_{-1/2}$$
$$(0{,}206) \qquad (0{,}413)$$

where

c = total private consumption;
$L_{B-1/2}$ = disposable income of private persons from wages and social payments (lagged half a year);
NL_B = disposable income of private persons excluding wages and social payments;
c_c = consumption credits;
i_m = private gross fixed investments, excluding dwellings;
v' = total expenditures minus increase of inventories minus exports of commodities minus non-material government consumption (output of commodities);
K = gross profit per unit production.

The standard deviations of the estimates are given in parentheses.

Step 1. As suggested in the previous sections, we prewhitened the individual input series which resulted in the input predicton-error (innovations) series given in Figs. 1–5.

Step 2. The prewhitened input series were used to estimate—as a first approximation—the $\{m_i\}$ from the finite lag approximation

$$y_t \approx \sum_{i=0}^{r} m_i \hat{\eta}_{t-i} + \epsilon_t,$$

where r is the maximum lag given the length of the time series. We have taken $r = 6$ for 32 observations.

Step 3. Form the finite Hankel with dimension $Np \times Nq$ where $N = (r + 1)/2$, $p = 1$ (the dimension of the output vector) and q the dimension of the input vector. Apply a singular value decomposition from which a balanced realization can be obtained which, in turn, can be used for a steady-state Kalman filter.

Step 4. (Re)-estimate the parameters of the Kalman filter by minimizing the prediction-error criterion using a non-linear minimization procedure. We used the balanced realization matrices $\hat{A}_b$, $\hat{B}_b$, $\hat{C}_b$ and $\hat{D} = \hat{m}_0$ together with $\hat{x}_0 = 0$ and $K = I$ as starting values, because the influence of the initial value $x_0 = \mu$ dies out quickly (see Picci, 1982).

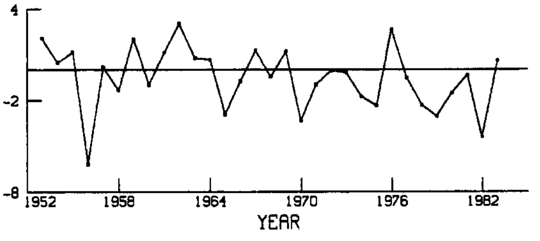

Fig. 1. Prewhitened input L_b.

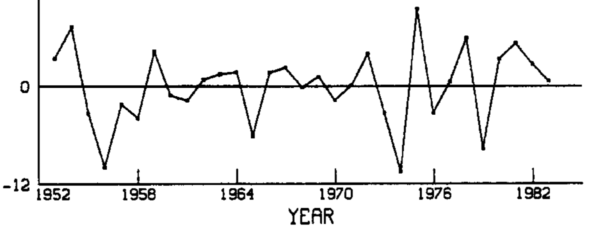

Fig. 2. Prewhitened input NL_b.

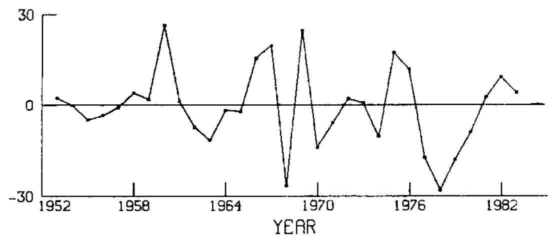

Fig. 3. Prewhitened input c_c.

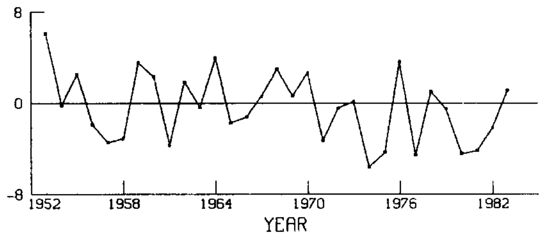

Fig. 4. Prewhitened input v'.

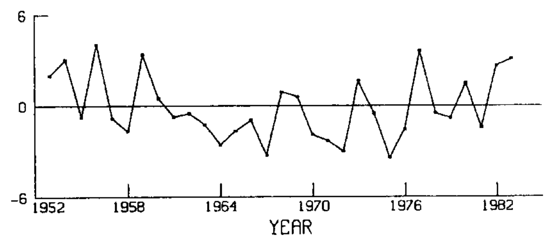

Fig. 5. Prewhitened input K.

The following singular values were obtained for the two macroeconomic equations (period 1952–1983): (a) for consumption

$$\sigma_1 = 2.3, \quad \sigma_2 = 1.2, \quad \sigma_3 = 0.7;$$

(b) for investment

$$\sigma_1 = 4.7, \quad \sigma_2 = 3.1, \quad \sigma_3 = 1.8.$$

The (ex-post) prediction for consumption (see Figs 6 and 7) were obtained with the following steady-state Kalman filter with $\dim(x_t) = 3$:

$$x_{t+1|t} = \hat{A}x_{t|t-1} + \hat{B}\hat{\eta}_t + \hat{K}e_t, \text{ initial condition } \hat{x}_0,$$
$$c_t = \hat{C}x_{t|t-1} + \hat{D}\hat{\eta}_t + e_t,$$

with input prediction $\hat{\eta}_{t|t-1} = 0$ for all t and

$$\hat{A} = \begin{bmatrix} 0.278 & -0.495 & -0.528 \\ 0.102 & 0.653 & -0.087 \\ -0.134 & -1.105 & -0.037 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0.635 & 0.405 & 0.128 \\ -1.404 & -0.518 & 0.079 \\ -0.323 & 0.052 & 0.09 \end{bmatrix},$$

$$\hat{C}' = (-0.476 \quad -0.571 \quad 0.526), \quad \hat{D}' = (-0.013 \quad -0.037 \quad 0.055),$$
$$\hat{K} = (0.996 \quad 2.019 \quad 3.101), \quad \hat{x}_0' = (-9.563 \quad 6.514 \quad 3.020)$$

and $\{\hat{\eta}_t\}$ is the prewhitened input series of L_B, NL_B and c_c. The (ex-post) predictions for investments

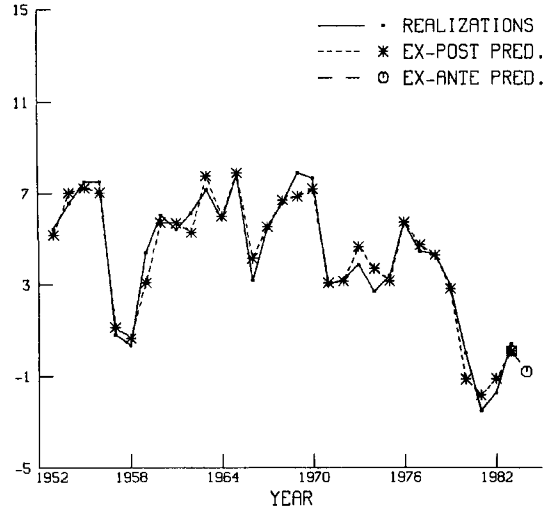

Fig. 6(a). Consumption [dim $(x_t) = 3$].

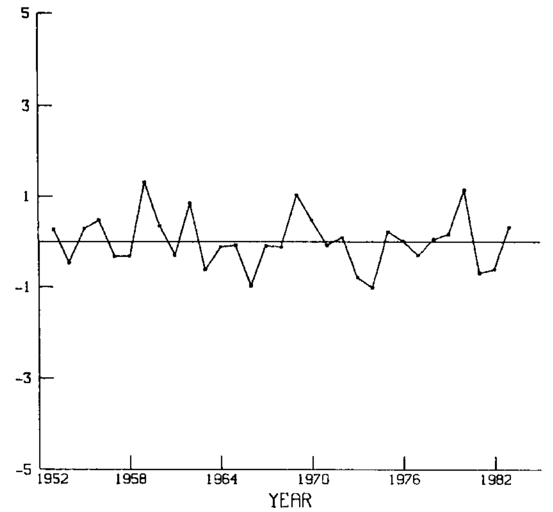

Fig. 6(b). Prediction-errors consumption [dim $(x_t) = 3$].

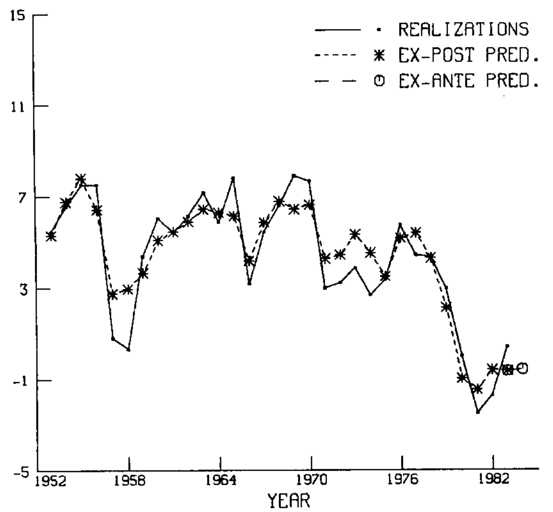

Fig. 7(a). Consumption [dim $(x_t) = 1$].

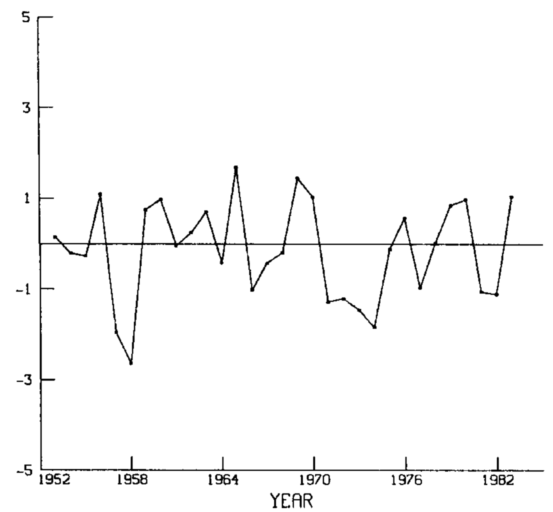

Fig. 7(b). Prediction-errors consumption [dim $(x_t) = 1$].

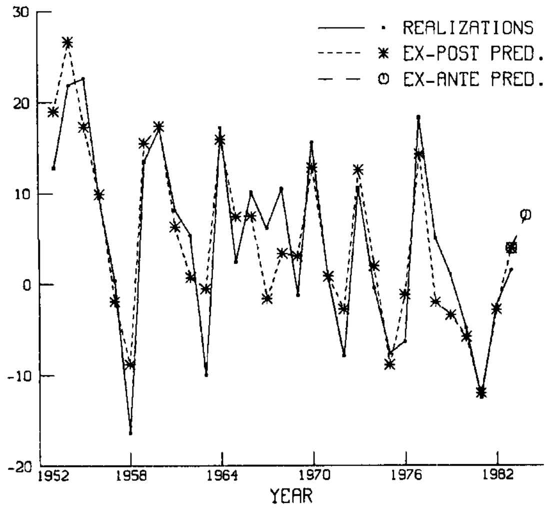

Fig. 8. Investments.

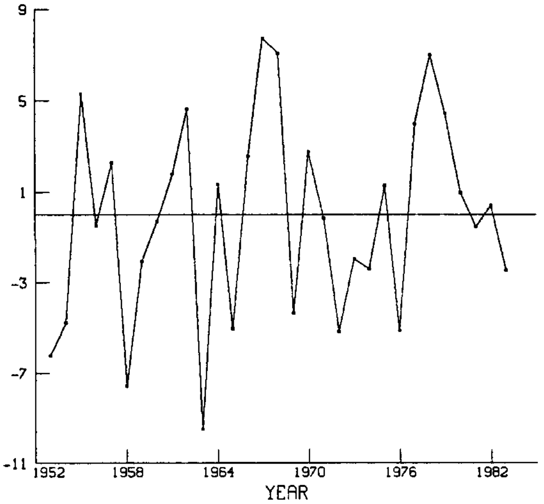

Fig. 9. Prediction-errors investments.

(see Figs 8 and 9), were obtained with the same steady-state Kalman filter but now with parameters

$$\hat{A} = \begin{bmatrix} 0.690 & -0.814 & -0.310 \\ -0.674 & -0.074 & -0.182 \\ 0.583 & -0.773 & -0.071 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 3.275 & 8.638 \\ 2.694 & 12.727 \\ 0.829 & 6.221 \end{bmatrix},$$

$$\hat{C}' = (0.860 \quad -0.202 \quad -0.766), \quad \hat{D}' = (1.337 \quad 1.480),$$

$$\hat{K}' = (0.098 \quad 0.706 \quad 1.363), \quad \hat{x}'_0 = (22.561 \quad -21.904 \quad -7.572).$$

In both cases the output prediction errors were tested on Gaussian white noise by using t-tests, Durbin–Watson tests, the white-noise test of Mehra and Peschon (1971), and the Kolmogorov–Smirnov test. In both cases there is evidence that the output prediction-error sequences are Gaussian white-noise. Decomposition of Theil's prediction measure U indicates no systematic component of the prediction errors (Theil, 1965).

5. CONCLUSIONS

In the foregoing a preliminary attempt has been made to the use system theoretic techniques to obtain quasi-balanced state space approximations from input–output records which can be used for prediction and/or control. The first results look promising although further research is needed, for example on prediction confidence intervals.

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